A Depth and norm

Since in our setting the length of sequences of enables transitions depends on the considered priority order, we define the depth and the norm of a process with respect to the empty order. The reason for this choice is twofold. Firstly, we notice that the depth defined with respect to the empty order is an upper bound for the depths defined with respect to any other priority order. Since for our purposes we will need to consider upper bounds for the depth of processes, and not the exact value of their depths, it is reasonable to consider directly the greatest of the depths. Notice that the norm defined with respect to the empty order is, dually, a lower bound for the norms defined with respect to the other priority orders. Secondly, this choice allows us to give alternative formulations of both measures by induction on the structure of processes.

Definition 8 (Depth and norm of processes). *The* depth *of a process is defined inductively on its structure by*
- $\mathrm{depth}\,(a) = 1$*;*
- $\mathrm{depth}\,(p_1 \cdot p_2) = \mathrm{depth}\,(p_1) + \mathrm{depth}\,(p_2)$*;*
- $\mathrm{depth}\,(p_1 + p_2) = \max\{\mathrm{depth}\,(p_1)\,, \mathrm{depth}\,(p_2)\}$*;*
- $\mathrm{depth}\,(\Theta(p)) = \mathrm{depth}\,(p)$.

Similarly, the norm *of process is defined inductively on its structure by*
- $\mathrm{norm}\,(a) = 1$*;*
- $\mathrm{norm}\,(p_1 \cdot p_2) = \mathrm{norm}\,(p_1) + \mathrm{norm}\,(p_2)$*;*
- $\mathrm{norm}\,(p_1 + p_2) = \min\{\mathrm{norm}\,(p_1)\,, \mathrm{norm}\,(p_2)\}$*;*
- $\mathrm{norm}\,(\Theta(p)) = \mathrm{norm}\,(p)$.

Both notions can be extended to process terms *by adding, respectively, the value of the depth and norm of a variable which are defined as* $\mathrm{depth}\,(x) = 1$ *and* $\mathrm{norm}\,(x) = 1$.

We remark that although variables cannot perform any transition, as one can easily see from the inference rules in Table 1, their depth, and norm, are set to 1, since the minimal closed instance of a variable with respect to these measures is as a constant in $\mathcal{A}$.

B Proofs of results in Section 3

B.1 Proof of Lemma 2

Proof of Lemma 2.

1. We proceed by induction over the derivation of the predicate $t \xrightarrow{x}_{>} \surd\!\!\surd$.
   - Base case: $t = x$ and $t \xrightarrow{x}_{>} \surd\!\!\surd$ is derived by rule (a_2) in Table 3. Hence the proof directly follows by $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\surd$.
   - Inductive step: $t = t_1 + t_2$ and $t \xrightarrow{x}_{>} \surd\!\!\surd$ is derived by either rule (a_8) in Table 3, and thus by $t_1 \xrightarrow{x}_{>} \surd\!\!\surd$, or its symmetric version on t_2. Assume wlog. that rule (a_8) in Table 3 was applied. Then by induction $t_1 \xrightarrow{x}_{>} \surd\!\!\surd$ and $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\surd$ imply $\sigma(t_1) \xrightarrow{a}_{>} \surd\!\!\surd$. Hence, the premise of rule (r_4) in Table 1 is satisfied and we can infer that $\sigma(t) \xrightarrow{a}_{>} \surd\!\!\surd$.

- Inductive step: $t = \Theta(u)$ and $t \xrightarrow{x}_{>} \surd\!\!\surd$ is derived by rule (a_{11}) in Table 3, and thus by $u \xrightarrow{x}_{>} \surd\!\!\surd$. By induction $u \xrightarrow{x}_{>} \surd\!\!\surd$ and $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\surd$ imply $\sigma(u) \xrightarrow{a}_{>} \surd\!\!\surd$. Since by the hypothesis action a has maximal priority with respect to $>$, the premises of rule (r_8) in Table 1 are satisfied and we can infer that $\sigma(t) \xrightarrow{a}_{>} \surd\!\!\surd$.

2. We proceed by induction over the derivation of the auxiliary transition $t \xrightarrow{x}_{>} t'$.
   - Base case: $t = t_1 \cdot t_2$ and $t \xrightarrow{x}_{>} t'$ is derived by rule (a_5) in Table 3, namely $t_1 \xrightarrow{x}_{>} \surd\!\!\surd$ and $t' = t_2$. By Lemma 2.1 we have that $t_1 \xrightarrow{x}_{>} \surd\!\!\surd$ and $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\surd$ imply that $\sigma(t_1) \xrightarrow{a}_{>} \surd\!\!\surd$. Hence, the premise of rule (r_2) in Table 1 is satisfied and we can infer that $\sigma(t) \xrightarrow{a}_{>} \sigma(t_2)$.
   - Inductive step: $t = t_1 \cdot t_2$ and $t \xrightarrow{x}_{>} t'$ is derived by rule (a_4) in Table 3, namely $t_1 \xrightarrow{x}_{>} t_1'$ and $t' = t_1' \cdot t_2$. By induction we have that $t_1 \xrightarrow{x}_{>} t_1'$ and $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\surd$ imply that $\sigma(t_1) \xrightarrow{a}_{>} \sigma(t_1')$. Hence, the premise of rule (r_3) in Table 1 is satisfied and we can infer that $\sigma(t) \xrightarrow{a}_{>} \sigma(t_1' \cdot t_2)$.
   - Inductive step: $t = t_1 + t_2$ and $t \xrightarrow{x}_{>} t'$ is derived either by rule (a_7) in Table 3, namely $t_1 \xrightarrow{x}_{>} t_1'$ and $t' = t_1'$, or by its symmetric version for t_2. Assume wlog. that rule (a_7) was applied. By induction we have that $t_1 \xrightarrow{x}_{>} t_1'$ and $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\surd$ imply that $\sigma(t_1) \xrightarrow{a}_{>} \sigma(t_1')$. Hence, the premise of rule (r_6) in Table 1 is satisfied and we can infer that $\sigma(t) \xrightarrow{a}_{>} \sigma(t_1')$.
   - Inductive step: $t = \Theta(u)$ and $t \xrightarrow{x}_{>} t'$ is derived by rule (a_{10}) in Table 3, namely $t_1 \xrightarrow{x}_{>} t_1'$ and $t' = \Theta(t_1')$. By induction we have that $t_1 \xrightarrow{x}_{>} t_1'$ and $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\surd$ imply that $\sigma(t_1) \xrightarrow{a}_{>} \sigma(t_1')$. Since by the hypothesis action a has maximal priority with respect to $>$, the premise of rule (r_9) in Table 1 is satisfied and we can infer that $\sigma(t) \xrightarrow{a}_{>} \sigma(\Theta(t_1'))$.

3. We proceed by induction over the derivation of the auxiliary transition $t \xrightarrow{x_s}_{>} c$.
   - Base case: $t = x$ and $t \xrightarrow{x_s}_{>} c$ is derived by rule (a_1) in Table 3, namely $c = x_d$. Hence the proof follows directly by $\sigma(x) \xrightarrow{a}_{>} p$.
   - Inductive step: $t = t_1 \cdot t_2$ and $t \xrightarrow{x_s}_{>} c$ is derived by rule (a_3) in Table 3, namely $t_1 \xrightarrow{x_s}_{>} c'$ and $c = c' \cdot t_2$. By induction we have that $t_1 \xrightarrow{x_s}_{>} c'$ and $\sigma(x) \xrightarrow{a}_{>} p$ imply $\sigma(t_1) \xrightarrow{a}_{>} p'$ for $p' = \sigma[x_d \mapsto p](c')$. Hence, by rule (r_3) in Table 1 we can infer that $\sigma(t) \xrightarrow{a}_{>} p' \cdot \sigma(t_2)$, with $p' \cdot \sigma(t_2) = \sigma[x_d \mapsto p](c' \cdot t_2)$.
   - Inductive step: $t = t_1 + t_2$ and $t \xrightarrow{x_s}_{>} c$ is derived either by rule (a_6) in Table 3, namely $t_1 \xrightarrow{x_s}_{>} c$, or by its symmetric version for t_2. Assume wlog. that (a_6) was applied. By induction we have that $t_1 \xrightarrow{x_s}_{>} c$ and $\sigma(x) \xrightarrow{a}_{>} p$ imply $\sigma(t_1) \xrightarrow{a}_{>} \sigma[x_d \mapsto p](c)$. Hence, by rule (r_6) in Table 1 we can infer that $\sigma(t) \xrightarrow{a}_{>} \sigma[x_d \mapsto p](c)$.
   - Inductive step: $t = \Theta(u)$ and $t \xrightarrow{x_s}_{>} \Theta(c)$ is derived by rule (a_9) in Table 3, namely $u \xrightarrow{x_s}_{>} c$. By induction we have that $u \xrightarrow{x_s}_{>} c$ and $\sigma(x) \xrightarrow{a}_{>} p$ imply $\sigma(u) \xrightarrow{a}_{>} \sigma[x_d \mapsto p](c)$. Since by the hypothesis action a has maximal priority with respect to $>$, by rule (r_9) in Table 1 we can infer that $\sigma(t) \xrightarrow{a}_{>} \sigma[x_d \mapsto p](\Theta(c))$.

$\square$

B.2 Proof of Lemma 3

Before proceeding to the proof, the provide an auxiliary technical Lemma, that will simplify our reasoning.

Lemma 7. *Let $a \in \mathcal{A}$ be maximal with respect to $>$, and let σ be a closed substitution. Consider a configuration c, and processes p, p' s.t. $p \xrightarrow{a}_{>} p'$. If c contains an occurrence of x_d, then $\sigma[x_d \mapsto p](c) \xrightarrow{a}_{>} \sigma[x_d \mapsto p'](c)$.*

Proof. We proceed by structural induction on c.

- Base case $c = t$: since c does not contain an occurrence of x_d, the lemma is vacuously true.
- Base case $c = x_d$: clearly, $\sigma[x_d \mapsto p](c) = p \xrightarrow{a}_{>} p' = \sigma[x_d \mapsto p'](c)$.
- Inductive step $c = c' \cdot t$: by induction over c' we obtain $\sigma[x_d \mapsto p](c') \xrightarrow{a}_{>} \sigma[x_d \mapsto p'](c')$. An application of rule (r_3) in Table 1 therefore gives

  $$\sigma[x_d \mapsto p](c) = \sigma[x_d \mapsto p](c') \cdot \sigma(t) \xrightarrow{a}_{>} \sigma[x_d \mapsto p'](c') \cdot \sigma(t) = \sigma[x_d \mapsto p'](c).$$

- Inductive step $c = \Theta(c')$: by induction over c' we have $\sigma[x_d \mapsto p](c') \xrightarrow{a}_{>} \sigma[x_d \mapsto p'](c')$. Since moreover a is maximal with respect to $>$, by applying rule (r_9) in Table 1 we obtain

  $$\sigma[x_d \mapsto p](c) = \sigma[x_d \mapsto p](\Theta(c')) \xrightarrow{a}_{>} \sigma[x_d \mapsto p'](\Theta(c')) = \sigma[x_d \mapsto p'](c).$$

□

Proof of Lemma 3. Note first of all that since $t \xrightarrow{x_s}_{>} c$, then c must contain an occurrence of x_d.

We proceed by induction on the derivation of $t \xrightarrow{x_s}_{>} c$, and for each case, we prove the statement by induction on n. However, since the base case of $n = 1$ is given by Lemma 2.3, we omit the details here. Furthermore, in each case we will use that fact that $\sigma(x) \to_{>}^{n} p$ implies that $\sigma(x) \to_{>}^{n-1} p' \to_{>} p$ for some process p'.

Rule (a_1): In this case we have $t = x$ and $c = x_d$. By induction hypothesis we get

$$\sigma(t) = \sigma(x) \to_{>}^{n-1} \sigma[x_d \mapsto p'](x_d) = p',$$

and we know that $p' \to_{>} p = \sigma[x_d \mapsto p](c)$, so we conclude that $\sigma(t) \to_{>}^{n} \sigma[x_d \mapsto p](c)$.

Rule (a_3): In this case we have $t = t_1 \cdot t_2$, $t_1 \xrightarrow{x_s}_{>} c'$, and $c = c' \cdot t_2$. By the induction hypothesis we get $\sigma(t_1) \to_{>}^{n-1} \sigma[x_d \mapsto p'](c')$, which, by rule (r_3), gives

$$\sigma(t) = \sigma(t_1) \cdot \sigma(t_2) \to_{>}^{n-1} \sigma[x_d \mapsto p'](c') \cdot \sigma(t_2) = \sigma[x_d \mapsto p'](c).$$

Since $p' \to_{>} p$, Lemma 7 gives $\sigma[x_d \mapsto p'](c) \to_{>} \sigma[x_d \mapsto p](c)$, so we conclude $\sigma(t) \to_{>}^{n} \sigma[x_d \mapsto p](c)$.

Rule (a_6): In this case we have $t = t_1 + t_2$ and $t_1 \xrightarrow{x_s}_> c$. The induction hypothesis gives $\sigma(t_1) \rightarrow_>^{n-1} \sigma[x_d \mapsto p'](c)$, so rule ($r_6$) and Lemma 7 together give

$$\sigma(t) \rightarrow_>^{n-1} \sigma[x_d \mapsto p'](c) \rightarrow_> \sigma[x_d \mapsto p](c).$$

A similar argument using rule (r_7) establishes the symmetric case.

Rule (a_9): In this case we have $t = \Theta(t')$, $t' \xrightarrow{x_s}_> c'$, and $c = \Theta(c')$. By the induction hypothesis we get $\sigma(t_1) \rightarrow_>^{n-1} \sigma[x_d \mapsto p'](c')$. Rule ($r_9$) and Lemma 7 then give

$$\sigma(t) \rightarrow_>^{n-1} \sigma[x_d \mapsto p'](\Theta(c')) = \sigma[x_d \mapsto p'](c) \rightarrow_> \sigma[x_d \mapsto p](c).$$

□

B.3 Proof of Lemma 4

Proof of Lemma 4. ($\Longrightarrow$) We proceed by structural induction on t in $x \lhd_\ell t$.

Case 1: We have $x \lhd_0 x$, so $x = t$, and hence rule (a_1) gives $t \xrightarrow{x_s}_> x_d = c_0$.

Case 2: We have $t = t_1 + t_2$ and either $x \lhd_\ell t_1$ or $x \lhd_\ell t_2$. If $x \lhd_\ell t_1$, then by induction hypothesis we get $t_1 \xrightarrow{x_s}_> c_\ell$, so rule ($a_6$) gives $t \xrightarrow{x_s}_> c_\ell$. If $x \lhd_\ell t_2$, we get the same by result by the symmetric version of (a_6).

Case 3: We have $t = t_1 \cdot t_2$ and $x \lhd_\ell t_1$, so by induction hypothesis we get $t_1 \xrightarrow{x_s}_> c'_\ell$. Rule ($a_3$) then gives $t \xrightarrow{x_s}_> c'_\ell \cdot t_2 = c_\ell$, which is of the correct form.

Case 4: We have $t = \Theta(t')$ and $x \lhd_{\ell-1} t'$. By induction hypothesis we get $t' \xrightarrow{x_s}_> c'_{\ell-1}$. Rule ($a_9$) gives $t \xrightarrow{x_s}_> \Theta(c'_{\ell-1}) = c_\ell$, which is of the correct form.

($\Longleftarrow$) The proof is by induction on the derivation of $t \xrightarrow{x_s}_> c_\ell$.

Rule (a_1): In this case we have $\ell = 0$, $t = x$, and $x \lhd_0 t$.

Rule (a_3): We have $t = t_1 \cdot t_2$ with $t_1 \xrightarrow{x_s}_> c'_\ell$. By induction hypothesis, this gives $x \lhd_\ell t_1$, which implies $x \lhd_\ell t_1 \cdot t_2 = t$.

Rule (a_6): We have $t = t_1 + t_2$ with $t_1 \xrightarrow{x_s}_> c_\ell$. The induction hypothesis then gives $x \lhd_\ell t_1$, which implies $x \lhd_\ell t_1 + t_2 = t$. The same argument holds for the symmetric version of (a_6).

Rule (a_9): We have $t = \Theta(t')$ with $t' \xrightarrow{x_s}_> c_{\ell-1}$. By induction hypothesis, this gives $x \lhd_{\ell-1} t'$, which implies $x \lhd_\ell \Theta(t') = t$.

□

B.4 Proof of Proposition 1

Proof of Proposition 1.

1. We proceed by induction over the derivation of $\sigma(t) \xrightarrow{a}_> \surd\!\!\surd$.
   - Base cases: $t = a$ and $t = x$. The proof for the former case follows directly by rule (r_1) in Table 1 and the latter directly by rule (a_2) in Table 3.
   - Inductive step $t = t_1 + t_2$ and $\sigma(t) \xrightarrow{a}_> \surd\!\!\surd$ is derived either by rule (r_4) in Table 1, and thus by $\sigma(t_1) \xrightarrow{a}_> \surd\!\!\surd$, or by rule ($r_5$) in Table 1, and thus by $\sigma(t_2) \xrightarrow{a}_> \surd\!\!\surd$. Assume wlog. that rule (r_4) was applied. By induction over $\sigma(t_1) \xrightarrow{a}_> \surd\!\!\surd$ we can distinguish two cases:

- $t_1 \xrightarrow{a}_{>} \surd\!\!\!\surd$. Then by rule (r_4) in Table 1 we derive that $t \xrightarrow{a}_{>} \surd\!\!\!\surd$.
- There is a variable x s.t. $t_1 \xrightarrow{x}_{>} \surd\!\!\!\surd$ and $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\!\surd$. Hence, by applying rule (a_8) in Table 3 we derive that, for the same variable x, $t \xrightarrow{x}_{>} \surd\!\!\!\surd$.

– Inductive step: $t = \Theta(u)$ and $\sigma(t) \xrightarrow{a}_{>} \surd\!\!\!\surd$ is derived by rule (r_8) in Table 1. This implies that $\sigma(u) \xrightarrow{a}_{>} \surd\!\!\!\surd$ and $\sigma(u) \stackrel{b}{\not\to}_{>}$ for all $b > a$. By induction over $\sigma(u) \xrightarrow{a}_{>} \surd\!\!\!\surd$ we can distinguish two cases:
- $u \xrightarrow{a}_{>} \surd\!\!\!\surd$. Since moreover from $\sigma(u) \stackrel{b}{\not\to}_{>}$ for all $b > a$ we can infer that $u \stackrel{b}{\not\to}_{>}$ for all such b, the premises of rule (r_8) in Table 1 are satisfied and we can derive that $t \xrightarrow{a}_{>} \surd\!\!\!\surd$.
- There is a variable x s.t. $u \xrightarrow{x}_{>} \surd\!\!\!\surd$ and $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\!\surd$. By applying rule (a_{11}) in Table 3 we derive that, for the same variable, $t \xrightarrow{x}_{>} \surd\!\!\!\surd$.

2. We proceed by induction over the derivation of $\sigma(t) \xrightarrow{a}_{>} p$.

– Base case: $t = x$. Then case (2c) is satisfied directly by rule (a_1) in Table 3.

– Inductive step: $t = t_1 \cdot t_2$. We can distinguish two cases:
- $\sigma(t) \xrightarrow{a}_{>} p$ is derived by rule (r_2) in Table 1, namely by $\sigma(t_1) \xrightarrow{a}_{>} \surd\!\!\!\surd$ and $p = \sigma(t_2)$. From $\sigma(t_1) \xrightarrow{a}_{>} \surd\!\!\!\surd$ and Proposition 1.1 we get that either $t_1 \xrightarrow{a}_{>} \surd\!\!\!\surd$ or there is a variable x s.t. $t_1 \xrightarrow{x}_{>} \surd\!\!\!\surd$ and $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\!\surd$. In the former case we can apply rule (r_2) in Table 1 and obtain $t \xrightarrow{a}_{>} t_2$ with $\sigma(t_2) = p$, thus case (2a) is satisfied. In the latter case we can apply rule (a_5) in Table 3 and obtain $t \xrightarrow{x}_{>} t_2$ which together with $\sigma(t_2) = p$ and $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\!\surd$ satisfies case (2b).
- $\sigma(t) \xrightarrow{a}_{>} p$ is derived by rule (r_3) in Table 1, namely by $\sigma(t_1) \xrightarrow{a}_{>} p_1$ with $p_1 = q \cdot \sigma(t_2)$. By induction over $\sigma(t_1) \xrightarrow{a}_{>} p_1$ we can distinguish three cases:
  - ∗ Case (2a) applies so that there is a process term t_1' s.t. $t_1 \xrightarrow{a}_{>} t_1'$ and $\sigma(t_1') = p_1$. Then, by rule (r_3) in Table 1 we infer that $t \xrightarrow{a}_{>} t_1' \cdot t_2$ with $\sigma(t_1') \cdot \sigma(t_2) = p$, and thus case (2a) is also satisfied by t.
  - ∗ Case (2b) applies so that there is a process term t_1' and a variable x s.t. $t_1 \xrightarrow{x}_{>} t_1'$, $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\!\surd$ and $\sigma(t_1') = p_1$. Then, by rule (a_4) in Table 3 we infer that $t \xrightarrow{x}_{>} t_1' \cdot t_2$ with $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\!\surd$ and $\sigma(t_1') \cdot \sigma(t_2) = p$, and thus case (2b) is also satisfied by t.
  - ∗ Case (2c) applies so that there are a variable x, a natural $l \in \mathbb{N}$ and a process s s.t. $t_1 \xrightarrow{x_s}_{>} \odot^l(x_d)$, $\sigma(x) \xrightarrow{a}_{>} q$ and $\odot^l(q) = p_1$. Then, by rule (a_3) in Table 3 we infer that $t \xrightarrow{x_s}_{>} \odot^l(x_d) \cdot t_2$. Hence case (2c) is also satisfied by t with respect to the configuration $\odot^l(x_d) \cdot t_2$, the variable x, the natural $l \in \mathbb{N}$ and the process q for which $\odot^l(q) \cdot \sigma(t_2) = p$.

– Inductive step: $t = t_1 + t_2$ and $\sigma(t) \xrightarrow{a}_{>} p$ is derived by the same transition performed either by $\sigma(t_1)$ or $\sigma(t_2)$, namely by applying either rule (r_6) or rule (r_7) in Table 1. Since induction applies to such a move taken by $\sigma(t_i)$ and in all the rules for nondeterministic choice in Tables 1 and 3 the moves of t_i are mimicked exactly by t, we can infer that each of the three cases of Proposition 1.2 holds for t whenever it holds for t_i.

- Inductive step: $t = \Theta(u)$ and $\sigma(t) \xrightarrow{a}_{>} p$ is derived by applying rule (r_9) in Table 1. This implies that $\sigma(u) \xrightarrow{a}_{>} p_1$, with $\Theta(p_1) = p$, and $\sigma(u) \not\xrightarrow{b}_{>}$ for all $b > a$. By induction over $\sigma(u) \xrightarrow{a}_{>} p_1$ we can distinguish three cases:
  - Case (2a) applies so that there is a process term u' s.t. $u \xrightarrow{a}_{>} u'$ and $\sigma(u') = p_1$. Moreover, we can also infer that $u \not\xrightarrow{b}_{>}$ for all $b > a$ because $\sigma(u) \not\xrightarrow{b}_{>}$. Then, by rule (r_9) in Table 1 we infer that $t \xrightarrow{a}_{>} \Theta(u')$ with $\sigma(\Theta(u')) = p$, and thus case (2a) is also satisfied by t.
  - Case (2b) applies so that there is a process term u' and a variable x s.t. $u \xrightarrow{x}_{>} u'$, $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\surd$ and $\sigma(u') = p_1$. Then, by rule (a_{10}) in Table 3 we infer that $t \xrightarrow{x}_{>} \Theta(u')$ with $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\surd$ and $\sigma(\Theta(u')) = p$, and thus case (2b) is also satisfied by t.
  - Case (2c) applies so that there are a variable x, a natural $l \in \mathbb{N}$ and a process q s.t. $u \xrightarrow{x_s}_{>} \odot^l(x_d)$, $\sigma(x) \xrightarrow{a}_{>} q$ and $\odot^l(q) = p_1$. Then, by rule (a_9) in Table 3 we infer that $t \xrightarrow{x_s}_{>} \Theta(\odot^l(x_d)) = \odot^{l+1}(x_d)$. Hence case (2c) is also satisfied by t with respect to the variable x, the natural $l+1$ and the process q for which $\odot^{l+1}(q) = p$.

□

B.5 Proof of Proposition 2

Before proceeding to the proof, we notice that by Lemma 7, if $p \xrightarrow{a}_{>} p'$ for some action a having (locally) maximal priority with respect to $>$, then the transition $\sigma[x_d \mapsto p](\odot^l(x_d)) \xrightarrow{a}_{>} \sigma[x_d \mapsto p'](\odot^l(x_d))$ is well defined. In this case, we abuse notation slightly and write directly $\odot^l(p) \xrightarrow{a}_{>} \odot^l(p')$.

Proof of Proposition 2. We proceed by induction over n.

- Base case $n = 1$. This directly follows by Proposition 1.2.
- Inductive step $n > 1$. $\sigma(t) \rightarrow^n_{>} p$ is equivalent to write $\sigma(t) \rightarrow_{>} p_1 \rightarrow^{n-1}_{>} p$, for some process p_1. We can assume wlog. that $\sigma(t) \xrightarrow{a}_{>} p_1$. Accordingly to Proposition 1.2, from $\sigma(t) \xrightarrow{a}_{>} p_1$ we can distinguish three cases:
  1. there is a process term t_1 s.t. $t \xrightarrow{a}_{>} t_1$ and $\sigma(t_1) = p_1$. Then by induction over $p_1 \rightarrow^{n-1}_{>} p$ we can distinguish two subcases:
     - there is $w_1 \in (\mathcal{A} \cup \mathcal{V})^*$ with $t_1 \xrightarrow{s_1 \dots s_h}_{>, w_1} t'$ with $|s_1 \dots s_h| = n-1$ and $\sigma(t') = p$. Then, the proof can be concluded by noticing that for the sequence $w = aw_1$ we get $t \xrightarrow{as_1 \dots s_h}_{>, w} t'$ with $|as_1 \dots s_h| = n$ and $\sigma(t') = p$.
     - there are $w_1 \in (\mathcal{A} \cup \mathcal{V})^*$, a variable y, a natural $l \in \mathbb{N}$ and a process q, such that $t_1 \xrightarrow{s_1 \dots s_h}_{>, w_1} t'$ with $|s_1 \dots s_h| = k < n-1$, $t' \xrightarrow{y_s}_{>} \odot^l(y_d)$, $\sigma(y) \rightarrow^{n-1-k}_{>} q$ and $\odot^l(q) = p$. Then, the proof can be concluded by noticing that for the sequence $w = aw_1$ we get $t \xrightarrow{as_1 \dots s_h}_{>, w} t'$ with $|as_1 \dots s_h| = k+1 < n$ and y, l, q behave as before.
  2. there are a process term t_1 and a variable x s.t. $t \xrightarrow{x}_{>} t_1$, $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\surd$ and $\sigma(t_1) = p_1$. Then by induction over $p_1 \rightarrow^{n-1}_{>} p$ we can distinguish two subcases:

- there is $w_1 \in (\mathcal{A} \cup \mathcal{V})^*$ with $t_1 \xrightarrow{s_1 \ldots s_h}_{>} t'$ with $|s_1 \ldots s_h| = n - 1$ and $\sigma(t') = p$. Then, the proof can be concluded by noticing that for the sequence $w = xw_1$ we get $t \xrightarrow{as_1 \ldots s_h}_{>,w} t'$ with $|as_1 \ldots s_h| = n$, as $|a| = 1$, and $\sigma(t') = p$.
- there are $w_1 \in (\mathcal{A} \cup \mathcal{V})^*$, a variable y, a natural $l \in \mathbb{N}$ and a process q, such that $t_1 \xrightarrow{s_1 \ldots s_h}_{>,w_1} t'$ with $|s_1 \ldots s_h| = k < n - 1$, $t' \xrightarrow{y_s}_{>} \odot^l(y_d)$, $\sigma(y) \rightarrow_{>}^{n-1-k} q$ and $\odot^l(q) = p$. Then, the proof can be concluded by noticing that, since $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\!/$ gives $|a| = 1$, for the sequence $w = xw_1$ we get $t \xrightarrow{as_1 \ldots s_h}_{>,w} t'$ with $|as_1 \ldots s_h| = k + 1 < n$ and c, x, q behave as before.

3. there are a variable x, a natural $m \in \mathbb{N}$ and a process p' s.t. $t \xrightarrow{x_s}_{>} \odot^m(x_d)$, $\sigma(x) \xrightarrow{a}_{>} p'$. More precisely, Lemma 3 allows us to distinguish two cases:
   - $\sigma(x) \rightarrow_{>}^{h} q$ for some $h \geq n$. In this case the thesis follows by considering $w = \emptyset$ and the process q' s.t. $\sigma(x) \rightarrow_{>}^{n} q'$ and $\odot^l(q') = p$.
   - $\sigma(x) \rightarrow_{>}^{k-1} q \rightarrow_{>} \surd\!\!\!/$ for some $k < n$. Notice that this implies that there is some string s_x with $|s_x| = k$ of actions that have been performed by $\sigma(x)$. Due to the structure of $\odot^l(x_d)$ we can infer that there are a natural $m' \in \mathbb{N}$ and a process term $t_1 = \underbrace{\Theta(\cdots \Theta}_{m' \text{ times}}(t'' \odot t_{m'+1}) \odot t_{m'}) \ldots) \odot u_1$ s.t. $\sigma(t) \rightarrow_{>}^{k} \sigma(t_1) = p_1$. Since then $p_1 \rightarrow_{>}^{n-k} p$, by induction we can distinguish two subcases:
     - there is $w_1 \in (\mathcal{A} \cup \mathcal{V})^*$ with $t_1 \xrightarrow{s_1 \ldots s_h}_{>,w_1} t'$ with $|s_1 \ldots s_h| = n - k$ and $\sigma(t') = p$. Then, the proof can be concluded by noticing that for the sequence $w = x_s w_1$ we get $t \xrightarrow{s_x s_1 \ldots s_h}_{>,w} t'$ with $|s_x s_1 \ldots s_h| = n$, as $|s_x| = k$, and $\sigma(t') = p$.
     - there are $w_1 \in (\mathcal{A} \cup \mathcal{V})^*$, a variable y, a natural $l \in \mathbb{N}$ and a process q', such that $t_1 \xrightarrow{s_1 \ldots s_h}_{>,w_1} t'$ with $|s_1 \ldots s_h| = j < n - k$, $t' \xrightarrow{y_s}_{>} \odot^l(y_d)$, $\sigma(y) \rightarrow_{>}^{n-k-j} q'$ and $\odot^l(q') = p$. Then, the proof can be concluded by noticing that, as $|s_x| = k$, for the sequence $w = x_s w_1$ we get $t \xrightarrow{s_x s_1 \ldots s_h}_{>,w} t'$ with $|s_x s_1 \ldots s_h| = k + j < n$ and y, l, q' behave as before.

□

B.6 Proof of Theorem 1

Proof of Theorem 1. Let $n \in \mathbb{N}$ be larger than the depths of t and u, and assume a priority order $>$ over $\mathcal{A}$ with $b > a$, with $a > c$ for any other possible action $c \in \mathcal{A}$. We define the family of closed substitutions $\{\sigma_i\}_{i \in \mathbb{N}}$ inductively as follows:

$$\sigma_0(y) = \begin{cases} a + b & \text{if } y = x \\ a & \text{otherwise.} \end{cases}$$

$$\sigma_i(y) = \begin{cases} a \cdot (\sigma_{i-1}(y) + a) & \text{if } y = x \\ a & \text{otherwise.} \end{cases}$$

Let $\sigma = \sigma_n$. Suppose that $t \rightarrow^k t'$, for some $k \in \mathbb{N}$. Since $\mathcal{A}^*(t) = \{a\}$, and all variables but x are mapped into a process that can only execute a, we can infer that there are process terms $t_0, \ldots, t_k$ s.t. $t = t_0 \xrightarrow{a} \ldots \xrightarrow{a} t_k = t'$. Moreover, as in all such terms t_i there is no occurrence of b, a is maximal with respect to $>$ on them, and thus by Lemma 1 and an easy induction over k, we obtain that $\sigma(t_0) \xrightarrow{a}^k \sigma(t_k)$, namely $\sigma(t) \xrightarrow{a}_>^k \sigma(t')$. Suppose now that $x \triangleleft_l t'$, for some $l \in \mathbb{N}$. By Lemma 4, $x \triangleleft_l t'$ implies that $t' \xrightarrow{x_s}_> \odot^l(x_d)$. By the choice of σ and $\mathcal{A}^*(t) = \{a\}$, we have that $\sigma(x) \xrightarrow{a}_>^n a + b$. Therefore, by Lemma 3 we obtain that $\sigma(t') \xrightarrow{a}_>^n \odot^l(a+b)$. By combining the two sequences of transitions, we get $\sigma(t) \xrightarrow{a}_>^{k+n} \odot^l(a+b)$. By the hypothesis we have $t \underline{\leftrightarrow}_* u$, which in particular implies $t \underline{\leftrightarrow}_> u$ and thus $\sigma(t) \underline{\leftrightarrow}_> \sigma(u)$. As $\underline{\leftrightarrow}_>$ is a bisimulation, we can infer that $\sigma(u) \xrightarrow{a}_>^{k+n} p$ for some process p with $\odot^l(a+b) \underline{\leftrightarrow}_> p$. As n is larger than the depth of u, by Proposition 2 there exist a process term u', a string w with strings $s_1, \ldots, s_h \in \{a\}^*$, a variable y, a natural number m and a process q such that $u \xrightarrow{s_1 \ldots s_h}_{>,w} u'$, $|s_1 \ldots s_h| = j < n$, $u' \xrightarrow{x_s}_> \odot^m(y_d)$, $\sigma(y) \rightarrow_>^{k+n-j} q$ and $p = \odot^m(q)$. Therefore: (i) by $k+n-j > 0$; (ii) by the choice of $>$ (which gives that the only possible transition enabled for $\odot^l(a+b)$ is a b-labeled move); (iii) by the choice of σ; (iv) by $\odot^l(a+b) \underline{\leftrightarrow}_> \odot^m(q)$; we can conclude that $y = x$, $j = k$ and $q = a + b$. Moreover, from $\odot^l(a+b) \underline{\leftrightarrow}_> \odot^m(a+b)$ and the choice of $>$, we obtain that $l = 0$ iff $m = 0$. □

C Proofs of results in Section 4

C.1 Proof of Lemma 5

Proof of Lemma 5. The proof is by induction on k. Note that $p \underline{\leftrightarrow}_* q$ in particular implies $p \underline{\leftrightarrow} q$.

Base case: If $k = 1$, assume that q is not uniformly 1-determinate. This means that either $|\mathcal{A}(q)| > 1$ or there exist q_1 and q_2 such that $q \rightarrow q_1$ and $q \rightarrow q_2$ but $q_1 \not\underline{\leftrightarrow}_\star q_2$, or $\mathrm{norm}\,(q_1) \neq 1$, or $\mathrm{norm}\,(q_2) \neq 1$.

If $|\mathcal{A}(q)| > 1$, then there are $a, b \in \mathcal{A}$ with $a \neq b$ such that $q \xrightarrow{a} q_a$ and $q \xrightarrow{b} q_b$ for some processes q_a and q_b. Since $p \underline{\leftrightarrow} q$, there must exist p_a and p_b such that $p \xrightarrow{a} p_a$ and $p \xrightarrow{b} p_b$, but this contradicts $|\mathcal{A}(p)| = 1$.

If $q_1 \not\underline{\leftrightarrow}_* q_2$, then $q_1 \not\underline{\leftrightarrow}_> q_2$ for some priority order $>$. Since we already know that $|\mathcal{A}(q)| = 1$, $q \rightarrow q_1$ and $q \rightarrow q_2$ implies $q \rightarrow_> q_1$ and $q \rightarrow_> q_2$. Hence there exist processes p_1 and p_2 such that $p \rightarrow_> p_1$ and $p \rightarrow_> p_2$ with $p_1 \underline{\leftrightarrow}_> q_1$ and $p_2 \underline{\leftrightarrow}_> q_2$. However, since p is uniformly 1-determinate, we know that $p_1 \underline{\leftrightarrow}_> p_2$, so $q_1 \underline{\leftrightarrow}_> q_2$, which is a contradiction.

If $\mathrm{norm}\,(q_1) \neq 1$, then we know from $p \underline{\leftrightarrow} q$ and $q \rightarrow q_1$ that $p \rightarrow p_1$ for some process p_1 with $p_1 \underline{\leftrightarrow} q_1$. But this implies $\mathrm{norm}\,(q_1) = \mathrm{norm}\,(p_1) = 1$, which is a contradiction. The argument for $\mathrm{norm}\,(q_2) \neq 1$ is similar.

Inductive step: Assume that q is uniformly k'-determinate for all $k' < k$. We now prove that q is also uniformly k-determinate. Assume towards a contradiction that q is not k-determinate. Then there must exist some q' such that $q \rightarrow^k q'$ and either

$|\mathcal{A}(q')| > 1$ or there are q_1 and q_2 such that $q' \to q_1$ and $q' \to q_2$, but either $q_1 \not\leftrightarrow_* q_2$, $\mathrm{norm}\,(q_1) \neq 1$, or $\mathrm{norm}\,(q_2) \neq 1$.

The cases of $|\mathcal{A}(q')| > 1$, $\mathrm{norm}\,(q_1) \neq 1$, and $\mathrm{norm}\,(q_2) \neq 1$ are essentially the same as for the base case, except that one first gets a process p' such that $p \to^k p'$, and then reasons as before on p'.

We now consider the case of $q_1 \not\leftrightarrow_* q_2$. This implies that $q_1 \not\leftrightarrow_> q_2$ for some priority order $>$. Since $p \leftrightarrow_* q$, we also get $p \leftrightarrow_> q$, and since we know that q is uniformly k'-determinate for every $k' < k$. $q \to^k q'$ implies $q \to_>^k q'$. Therefore there exists a process p' such that $p \to_>^k p'$ and $p' \leftrightarrow_> q'$. Since we already know that $|\mathcal{A}(q')| = 1$, $q' \to q_1$ and $q' \to q_2$ implies $q' \to_> q_1$ and $q' \to_> q_2$. Hence there exist p_1 and p_2 such that $p' \to_> p_1$ and $p' \to_> p_2$ as well as $p_1 \leftrightarrow_> q_1$ and $p_2 \leftrightarrow_> q_2$. However, since p is uniformly k-determinate, we know that $p_1 \leftrightarrow_> p_2$, so we get $q_1 \leftrightarrow_> q_2$, which contradicts our assumption.

□

C.2 Proof of Proposition 3

Proof of Proposition 3. Since our notion of uniformly k-determinate implies that of k-determinate in [3], Lemma 18 of that paper gives the result. □

D Proofs of results in Section 5

D.1 Proof of Proposition 4

Proof of Proposition 4. Notice that it is enough to prove that $A_n(\Theta(a + b)) \approx P_n$, since then

$$A_n(\Theta(a + b)) \approx P_n \implies P_n + A_n(\Theta(a + b)) \approx P_n + P_n \approx P_n.$$

Let $>$ be an arbitrary preorder. We now proceed by a case analysis on the behaviour of $A_n(\Theta(a + b))$ with respect to $>$.

- If $a > b$, then $A_n(\Theta(a + b)) \leftrightarrow_> A_n(a)$.
- If $b > a$, then $A_n(\Theta(a + b)) \leftrightarrow_> A_n(b)$.
- If a and b are incomparable, then $A_n(\Theta(a + b)) \leftrightarrow_> A_n(a + b)$.

In any case, we conclude that $A_n(\Theta(a + b)) \leftrightarrow_> P_n$. □

D.2 Proof of Lemma 6

Before proceeding to the proof, we recall a preliminary result on $\leftrightarrow_>$. For a given priority order $>$, the bisimulation equivalence $\leftrightarrow_>$ behaves like a classic bisimulation and therefore the following Lemma holds. (The same result on BCCSP processes was provided as Proposition 9 in [3]).

Lemma 8. *Consider processes p, q, assume $p \leftrightarrow_> q$, for some priority order $>$ over $\mathcal{A}$, and let $k \in \mathbb{N}$. Then:*

1. *For every process p' s.t. $p \to_{>}^{k} p'$, there is a process q' s.t. $q \to_{>}^{k} q'$ and $p' \leftrightarrow_{>} q'$.*
2. $\mathcal{A}_{>}^{k}(p) = \mathcal{A}_{>}^{k}(q)$ *so, in particular,* $\mathcal{A}_{>}^{1}(p) = \mathcal{A}_{>}^{1}(q)$.

Proof of Lemma 6. We first prove that $\mathcal{A}^k(p) = \{a\}$ for $0 \leq k < n$. Assume $p \sqsubseteq_* P_n$, which means that $p + r \leftrightarrow_* P_n$ for some r, which in particular implies that $p + r \leftrightarrow P_n$. By Lemma 8, we infer $\mathcal{A}^k(p + r) = \mathcal{A}^k(P_n) = \{a\}$. Since, moreover, $\mathcal{A}^k(p) \subseteq \mathcal{A}^k(p + r)$, we get $\mathcal{A}^k(p) = \{a\}$.

We proceed by contradiction. Let $1 \leq k < n$ be the least number such that p is not uniformly k-determinate. Then there exist processes p', p_1, and p_2 such that $p \to^k p'$, $p' \to p_1$, and $p' \to p_2$, and $p_1 \not\leftrightarrow_* p_2$, or $\mathrm{norm}\,(p_1) \neq 1$, or $\mathrm{norm}\,(p_2) \neq 1$.

If $\mathrm{norm}\,(p_1) \neq 1$, then $p \to^k p'$ and $p' \to p_1$, so there exists P'_n and P''_n such that $P_n \to^k P'_n$ and $P'_n \to P''_n$ with $p_1 \leftrightarrow P''_n$. But then $\mathrm{norm}\,(p_1) = \mathrm{norm}\,(P''_n) = 1$, which is a contradiction. A similar argument holds when $\mathrm{norm}\,(p_2) \neq 1$.

If $p_1 \not\leftrightarrow_* p_2$, then $p_1 \not\leftrightarrow_> p_2$ for some specific priority order $>$. Notice that since $|\mathcal{A}^i(p)| = \{a\}$ for all $0 \leq i < n$, we get that $p \to^k p'$, $p' \to p_1$, and $p' \to p_2$ implies $p \to_{>}^{k} p'$, $p' \to_> p_1$, and $p' \to_> p_2$. Since $p + r \leftrightarrow_> P_n$ for some r, there exist P'_n, P''_n, and P'''_n such that $P_n \to_{>}^{k} P'_n$, $P'_n \to_> P''_n$, and $P'_n \to_> P'''_n$ with $p_1 \leftrightarrow_> P''_n$ and $p_2 \leftrightarrow_> P''_n$. Since $\mathrm{norm}\,(p_1) = 1 = \mathrm{norm}\,(p_2)$, we also get $\mathrm{norm}\,(P''_n) = 1 = \mathrm{norm}\,(P'''_n)$. However, we see from the definition of P_n that P'_n has a unique successor with norm 1. Hence it follows that $P''_n = P'''_n$, so $p_1 \leftrightarrow_> P''_n = P'''_n \leftrightarrow_> p_2$, which contradicts $p_1 \not\leftrightarrow_> p_2$. □

D.3 Proof of Proposition 5

Proof of Proposition 5. We start by noticing that since $\sigma(t)$ is uniformly Θ-n-dependent, by Definition 4 there are processes $p_0, \ldots, p_n$ s.t. $\sigma(t) = p_0 \to \ldots \to p_n$, $\mathrm{norm}\,(p_i) = 1$ for all $i = 0, \ldots, n-1$, and p_n is Θ-dependent. Since, moreover, we have $\mathrm{depth}\,(t) < n$, by Proposition 2 there are a process term t' and a string w s.t. $t \xrightarrow{s_1 \ldots s_h}_w t'$ and there are a variable x, an $l \in \mathbb{N}$ and a process q s.t. $t' \xrightarrow{x_s} \odot^l(x_d)$, $\sigma(x) \to^{n-k} q$, and $\odot^l(q) = p_n$.

Notice that, by Lemma 4, $t' \xrightarrow{x_s} \odot^l(x_d)$ is the same as $x \triangleleft_l t'$. Since, moreover, p_n is Θ-dependent, it must be the case that $|\mathcal{A}| > 1$. We can then apply Theorem 1, thus obtaining that there are a process term u' and an $m \in \mathbb{N}$ s.t. $u \to^k u'$ and $x \triangleleft_m u'$. Using again Lemma 4, $x \triangleleft_m u'$ is the same as $u' \xrightarrow{x_s} \odot^m(x_d)$. Notice that $\sigma(u) \sqsubseteq_* P_n$ implies that $\mathcal{A}^{n-1}(\sigma(u)) = \{a\}$. Hence, we have that a is locally maximal with respect to any priority order. Thus, from $\sigma(x) \xrightarrow{a}^{n-k} q$ and $u' \xrightarrow{x_s} \odot^m(x_d)$, Lemma 3 implies $\sigma(u') \xrightarrow{a}^{n-k} \odot^m(q)$. Hence we can infer that there are processes $q_0, \ldots, q_n$ s.t. $\sigma(u) = q_0 \to \ldots \to q_n = \odot^m(q)$. As p_n is Θ-dependent, $l > 0$ and thus, by Theorem 1, we can infer that $m > 0$, so that also $\odot^m(q)$ is Θ-dependent.

To conclude, we need to show that $\mathrm{norm}\,(q_i) = 1$ for each $i = 0, \ldots, n-1$. First of all we notice that, since $\sigma(t) \leftrightarrow_* \sigma(u)$ and $\mathrm{norm}\,(\sigma(t)) = 1$, then $\mathrm{norm}\,(\sigma(u)) = \mathrm{norm}\,(q_0) = 1$. Moreover, since by the hypothesis $\sigma(t)$ is uniformly k-determinate for all $1 \leq k < n$, by Lemma 5 we infer that also $\sigma(u)$ is uniformly k-determinate for the same values of k, and thus $\mathrm{norm}\,(q_i) = 1$ for all $i = 1, \ldots, n-1$ is guaranteed by Definition 4. We can therefore conclude that $\sigma(u)$ is uniformly Θ-n-dependent. □

$$(e_1)\,\frac{}{t \approx t} \qquad (e_2)\,\frac{t \approx u}{u \approx t} \qquad (e_3)\,\frac{t \approx u \quad u \approx v}{t \approx v} \qquad (e_4)\,\frac{t \approx u}{\sigma(t) \approx \sigma(u)}$$

$$(e_5)\,\frac{t_1 \approx u_1 \quad t_2 \approx u_2}{t_1 \cdot t_2 \approx u_1 \cdot u_2} \qquad (e_6)\,\frac{t_1 \approx u_1 \quad t_2 \approx u_2}{t_1 + t_2 \approx u_1 + u_2} \qquad (e_7)\,\frac{t \approx u}{\Theta(t) \approx \Theta(u)}$$

Table 4: Rules of equational logic over BPA$_\Theta$.

D.4 Proof of Theorem 2

Before proceeding to the proof of our main result, we report, in Table 4, the rules of equaitonal logic over BPA$_\Theta$. As in operational semantics, they allow us to infer equations by proceeding inductively over the structure of terms. Let E be a sound set of axioms. Rules (e_1)-(e_4) are common for all process languages and they ensure that E is closed with respect to reflexivity, symmetry, transitivity and substitution, respectively. Rules (e_5)-(e_7) are tailored for BPA$_\Theta$ and they ensure the closure of E under BPA$_\Theta$ contexts. They are therefore referred to as the *congruence rules*. Briefly, rule (e_5) is the rule for sequential composition and it states that whenever $E \vdash t_1 \approx u_1$ and $E \vdash t_2 \approx u_2$, then we can infer $E \vdash t_1 \cdot u_1 \approx t_2 \cdot u_2$. Rule ($e_6$) deals with the nondeterministic choice operator in a similar way and rule (e_7) ensures that the priority operator preserves the equivalence of terms.

As elsewhere in the literature, we assume without loss of generality that for each axiom in E also the symmetric counterpart is in E, so that the symmetry rule is not necessary in the proofs, and that substitutions rules are always applied first in equational proofs, which means that the substitution rule $\frac{t \approx u}{\sigma(t) \approx \sigma(u)}$ may only be used over axioms $t \approx u$ in E. If this is the case, then $\sigma(t) \approx \sigma(u)$ is called a *substitution instance* of the axiom.

Moreover, we will make use of the following technical result from [3].

Lemma 9 ([3, Lemma 14]). *If $p \underline{\leftrightarrow}_* q$ and p is Θ-dependent, then so is q.*

We are now ready to prove our main result.

Proof of Theorem 2. As briefly discussed in Section 2, without loss of generality, we can disregard the symmetry rule in our inductive proof below by assuming that $u \approx t \in E$ whenever $t \approx u \in E$. Furthermore, we can assume that all applications of the substitution rule in derivations have a process equation from E as premise. This means that we only need to consider a new rule stating that all substitution instances of process equations in E are derivable, rather than considering the axiom rule — which states that all process equations in E are derivable —, and the substitution rule — which states that if a process equation is derivable, then so are all its substitution instances — separately.

We will now present the inductive argument over the number of steps in a proof of an equation $p \approx q$ from E. We proceed by a case analysis on the last rule applied to obtain $E \vdash p \approx q$.

Case 1: *reflexivity and transitivity.* In these cases, the proof follows immediately or by the induction hypothesis in a straightforward manner.

Case 2: *variable substitution.* Assume that $E \vdash p \approx q$ is the result of a closed substitution instance of an open process equation $t \approx u \in E$, namely there exists a substitution σ such that $\sigma(t) = p$ and $\sigma(u) = q$. Since $t \approx u \in E$, we have that $\mathrm{depth}(t), \mathrm{depth}(u) < n$. Moreover, from $p, q \sqsubseteq_* P_n$ it follows that $\mathcal{A}^*(p) = \mathcal{A}^*(q) = \{a\}$ and that, by Lemma 6, p and q are uniformly k-determinate for all $k \in \{1, \dots n-1\}$. Hence by Proposition 5, we can conclude that if p is uniformly Θ-n-dependent, then so is q.

Case 3: *congruence rule.* We can distinguish three cases:

– The last rule applied in $E \vdash p \approx q$ is the congruence rule for the nondeterministic choice $+$. Then there exist closed process terms p_1, p_2, q_1 and q_2 such that $p = p_1 + p_2$, $q = q_1 + q_2$, $E \vdash p_1 \approx q_1$ and $E \vdash p_2 \approx q_2$ by shorter proofs. Since p is uniformly Θ-n-dependent, there must exist a process p' such that $p \to^n p'$, where p' is Θ-dependent and every process along the transitions from p to p' has norm 1. We can distinguish four possible subcases, regarding how such property is derived:
  1. p_1 is Θ-n-dependent.
  2. p_2 is Θ-n-dependent.
  3. $\mathrm{norm}(p_2) = 1$, $\mathrm{norm}(p_1) \neq 1$, and there are processes $p_1^1, \dots, p_1^n$ such that $p_1 \to p_1^1 \to p_1^n = p'$ and p_1^n is Θ-dependent.
  4. $\mathrm{norm}(p_1) = 1$, $\mathrm{norm}(p_2) \neq 1$, and there are processes $p_2^1, \dots, p_2^n$ such that $p_2 \to p_2^1 \to p_2^n = p'$ and p_2^n is Θ-dependent.

  In cases (1) and (2) we can immediately apply the induction hypothesis obtaining, respectively, that either q_1 or q_2 is Θ-n-dependent, and thus that q is Θ-n-dependent as well.

  The cases (3) and (4) require more attention. We detail only the proof for case (3), since the one for case (4) is symmetric. Firstly, we notice that since $p, q \sqsubseteq_* P_n$ then by Lemma 6 both p and q are uniformly k-determinate for all $k \in \{1, \dots, n-1\}$. This implies that p_1 is uniformly k-determinate for the same values of k. Moreover, as $E \vdash p_1 \approx q_1$ gives $p_1 \underline{\leftrightarrow}_* q_1$ and $\mathrm{depth}(p_1) = n$, by Lemma 5 we obtain that also q_1 is uniformly k-determinate for $k \in \{1, \dots, n-1\}$. Then, by Proposition 3 we can infer that there is a a process q_1^n such that $q_1 \to^n q_1^n$ and $q_1^n \underline{\leftrightarrow}_* p_1^n$, which, by Lemma 9, implies that q_1^n is Θ-dependent. Furthermore, uniform k-determinacy ensures that all the processes $q_1^1, \dots, q_1^{n-1}$ in the sequence $q_1 \to q_1^1 \to \dots \to q_1^{n-1} \to q_1^n$ have norm 1. Finally, we notice that since $\mathrm{norm}(p_2) = 1$ and $E \vdash p_2 \approx q_2$ implies $p_2 \underline{\leftrightarrow}_* q_2$, we can infer that $\mathrm{norm}(q_2) = 1$. By combining the properties of q_1 and q_2, we can conclude that $q = q_1 + q_2$ is uniformly Θ-n-dependent.

– The last rule applied in $E \vdash p \approx q$ is the congruence rule for the sequential composition. This means that $p = p_1 \cdot p_2$, $q = q_1 \cdot q_2$, $E \vdash p_1 \approx q_1$ and $E \vdash p_2 \approx q_2$ by shorter proofs. This case is vacuous, as $\mathrm{norm}(p) \geq 2$ and therefore p cannot be uniformly Θ-n-dependent.

– The last rule applied in $E \vdash p \approx q$ is the congruence rule for the priority operator Θ. Then there exist p' and q' such that $p = \Theta(p')$, $q = \Theta(q')$, and $E \vdash p' \approx q'$ by a shorter proof. Since p is uniformly Θ-n-dependent, there exists a sequence of processes $p = \Theta(p') \to \Theta(p_1) \to \cdots \to \Theta(p_{n-1}) \to \Theta(p_n)$ such that $\mathrm{norm}\,(\Theta(p_1)) = \ldots \mathrm{norm}\,(\Theta(p_{n-1})) = 1$ and $\Theta(p_n)$ is Θ-dependent. Note that, since $\Theta(p_n)$ is Θ-dependent, $|\mathcal{A}(p_n)\,| \geq 2$. Moreover, from the operational rules for Θ, $p' \to p_1 \to \cdots \to p_{n-1} \to p_n$ and from the definition of norm, $\mathrm{norm}\,(p_1) = \cdots = \mathrm{norm}\,(p_n) = 1$. From $E \vdash p' \approx q'$, we derive that $p' \underline{\leftrightarrow}_* q'$. Hence, $p' \underline{\leftrightarrow} q'$ holds and therefore we get a sequence $q' \to q_1 \to \cdots \to q_n$ such that $p_n \underline{\leftrightarrow} q_n$, which implies that $|\mathcal{A}(q_n)\,| \geq 2$. Thus, we infer $q = \Theta(q') \to \Theta(q_1) \to \cdots \to \Theta(q_n)$ and, since $|\mathcal{A}(q_n)\,| \geq 2$, $\Theta(q_n)$ is Θ-dependent. It remains to show that $\mathrm{norm}\,(\Theta(q')) = \mathrm{norm}\,(\Theta(q_i)) = 1$ for each $i \in \{1, \ldots, n-1\}$. As $q \sqsubseteq_* P_n$, by Lemma 6 we gather that q is uniformly k-determinate for all $1 \leq k < n$, from which it follows that $\mathrm{norm}\,(\Theta(q_i)) = 1$ for all $i \in \{1, \ldots, n-1\}$. Since, moreover, $p \underline{\leftrightarrow}_* q$ and $\mathrm{norm}\,(p) = 1$, we get $\mathrm{norm}\,(q) = 1$ and we conclude that q is Θ-n-dependent.

□